Applying the Schwarz inequality to the second sum (8)
on the right, we obtain

$$\sum_{n=1}^{\infty}(s_n+t_n)^2 \le \sum_{n=1}^{\infty} s_n^2 + 2\left(\sum_{n=1}^{\infty} s_n^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} t_n^2\right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} t_n^2$$

$$\sum_{n=1}^{\infty}(s_n+t_n)^2 \le \left[\left(\sum_{n=1}^{\infty} s_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} t_n^2\right)^{\frac{1}{2}}\right]^2$$

Taking square root on both side.

$$\left[\sum_{n=1}^{\infty}(s_n+t_n)^2\right]^{\frac{1}{2}} \le \left(\sum_{n=1}^{\infty} s_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} t_n^2\right)^{\frac{1}{2}}$$

Hence proved.

Chapter 4 Limits and metric spaces.

4.1 Limits of a function on the real line.

4.1A Definition We say that $f(x)$ approaches to L (where $L \in \mathbb{R}$) as x approaches to a, if given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad (0 < |x-a| < \delta).$$

In this case we write $\lim_{x \to a} f(x) = L$ (or) $f(x) \to L$ as $x \to a$.

Theorem 4.1B:
If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $f(x)+g(x)$ has a limit as $x \to a$ and $\lim_{x\to a}(f(x)+g(x)) = L+M$.

Proof: Given $\lim_{x\to a} f(x) = L$ therefore by definition, given $\varepsilon > 0$ we must find $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{2} \qquad (0 < |x-a| < \delta_1) \quad \text{——} \ (1)$$

also given $\lim_{x\to a} g(x) = M$, By defn given $\epsilon > 0$ there exists $\delta_2 > 0$ such that

$$|g(x) - M| < \epsilon/2 \qquad 0 < |x-a| < \delta_2 \quad \text{——— (2)}$$

let $\delta = \min\{\delta_1, \delta_2\}$ and if $0 < |x-a| < \delta$, then

$$|f(x) - L| < \epsilon/2 \;, \quad |g(x) - M| < \epsilon/2$$

now if $0 < |x-a| < \delta$

$$|(f(x)+g(x)) - (L+M)| = |(f(x)-L) + (g(x)-M)|$$

$$\le |f(x)-L| + |g(x)-M|$$

$$< \epsilon/2 + \epsilon/2$$

$$|(f(x)+g(x)) - (L+M)| < \epsilon$$

$$\Rightarrow \lim_{x\to a}(f(x)+g(x)) = L+M.$$ Hence proved

Theorem 4.1 C If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

a) $\lim_{x\to a}[f(x) - g(x)] = L - M$.

b) $\lim_{x\to a}(f(x)g(x)) = LM$ and if $M \neq 0$

c) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$.

Proof(a) given $\lim_{x\to a} f(x) = L$ by definition, given $\epsilon > 0$ there exists $\delta_1 > 0$ such that

$$|f(x) - L| < \epsilon/2 \qquad 0 < |x-a| < \delta_1 \quad \text{——— (1)}$$

Also given $\lim_{x\to a} g(x) = M$, therefore by defn.

given $\varepsilon_1 > 0$ there exists $\delta_2 > 0$ such that

$$|g(x) - M| < \frac{\varepsilon_1}{2} \quad 0 < |x-a| < \delta_2 \quad \text{——— } (2)$$

let $\delta = \text{Min}\{\delta_1, \delta_2\}$

$\Rightarrow$ if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon_1/2$ and

$|g(x) - M| < \varepsilon_1/2$

now if $0 < |x-a| < \delta$

Consider

$$\begin{aligned}
\left|(f(x) - g(x)) - (L - M)\right| &= \left|(f(x) - L) + (M - g(x))\right| \\
&\le |f(x) - L| + |M - g(x)| \\
&\le |f(x) - L| + |g(x) - M| \\
&< \varepsilon_1/2 + \varepsilon_1/2
\end{aligned}$$

$$\left|(f(x) - g(x)) - (L - M)\right| < \varepsilon_1 \quad \text{if } 0 < |x-a| < \delta$$

$$\Rightarrow \lim_{x\to a} (f(x) - g(x)) = L - M.$$

Proof (b) and (c) are already proved in sequence similar proof so try yourself.

4.1D <u>Definition</u> we say that $f(x)$ approaches to L as x approaches infinity if given $\varepsilon > 0$ there exists $M \in \mathbb{R}$ such that

$$|f(x) - L| < \varepsilon \quad (x > M).$$

we can write $\lim_{x \to \infty} f(x) = L$ (or) $f(x) \to L$ as $x \to \infty$.

4.1E <u>Defn</u> : Right hand limit of f at a.

We say that $f(x)$ approaches to L as x approaches to a from the right if given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad (a < x < a + \delta).$$

In this case we write

$$\lim_{x \to a+} f(x) = L$$

L is called the right-hand limit of f at a.

<u>Defn</u> M is left hand limit of f at a.

we say that $f(x)$ approaches M as x approaches a from the left, if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - M| < \varepsilon$ when $(a - \delta < x < a)$

in this case we can write

$$\lim_{x \to a-} f(x) = M$$

The number M is called the left-hand limit of f at a.
